Wednesday 17 May 2023

— Day 1 —

1. In a triangle $ABC$ with $AB < AC$, $D$ is a point on segment $AC$ such that $BD = CD$. A line parallel to $BD$ meets segment $BC$ at $E$ and line $AB$ at $F$. Point $G$ is the intersection of $AE$ and $BD$.
   Show that $\angle BCG = \angle BCF$.

2. Find all positive integers $m$ and $n$ with no common divisor greater than 1 such that $m^3 + n^3$ divides $m^2 + 20mn + n^2$.

3. Consider a sequence of real numbers defined by:
   \[
   x_1 = c \\
   x_{n+1} = cx_n + \sqrt{c^2 - 1} \sqrt{x_n^2 - 1} \quad \text{for all } n \geq 1.
   \]
   Show that if $c$ is a positive integer, then $x_n$ is an integer for all $n \geq 1$. 

Time: 4 hours and 30 minutes
Each problem is worth 7 points
4. Manzi has $n$ stamps and an album with 10 pages. He distributes the $n$ stamps in the album such that each page has a distinct number of stamps. He finds that, no matter how he does this, there is always a set of 4 pages such that the total number of stamps in these 4 pages, is at least $n/2$. Determine the maximum possible value of $n$.

5. Let $a$ and $b$ be real numbers with $a \neq 0$. Let

$$P(x) = ax^4 - 4ax^3 + (5a + b)x^2 - 4bx + b.$$ 

Show that all roots of $P(x)$ are real and positive if and only if $a = b$. 

6. Let $ABC$ be an acute triangle with $AB < AC$. Let $D$, $E$, and $F$ be the feet of the perpendiculars from $A$, $B$, and $C$ to the opposite sides, respectively. Let $P$ be the foot of the perpendicular from $F$ to line $DE$. Line $FP$ and the circumcircle of triangle $BDF$ meet again at $Q$. Show that $\angle PBQ = \angle PAD$.

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