PAMO 2022
Benguerir, Morocco

Saturday, June 25, 2022

— Day 1 —

1. Let ABC be a triangle with $\angle ABC \neq 90^\circ$, and AB its shortest side. Let H be the orthocenter of ABC. Let Γ be the circle with center B and radius BA. Let D be the second point where the line CA meets Γ. Let E be the second point where Γ meets the circumcircle of the triangle BCD. Let F be the intersection point of the lines DE and BH.

Prove that the line BD is tangent to the circumcircle of the triangle DFH.

2. Find all 3-tuples (a, b, c) of positive integers, with $a \geqslant b \geqslant c$, such that $a^2 + 3b$, $b^2 + 3c$ and $c^2 + 3a$ are all squares.

3. Let n be a positive integer and $a_1, a_2, \ldots, a_{2n}$ be a sequence of positive real numbers whose product is equal to 2. For $k = 1, 2, \ldots, 2n$, set $a_{2n+k} = a_k$ and define

$$A_k = \frac{1 + a_k + a_k a_{k+1} + \cdots + a_k a_{k+1} \ldots a_{k+n-2}}{1 + a_k + a_k a_{k+1} + \cdots + a_k a_{k+1} \ldots a_{k+2n-2}}.$$

Suppose that $A_1, A_2, \ldots, A_{2n}$ are pairwise distinct; show that exactly half of them are less than $\sqrt{2} - 1$.

Time: 4 hours and 30 minutes
Each problem is worth 7 points

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Sunday, June 26, 2022

— Day 2 —

4. Find all functions f and g defined from $\mathbb{R}_{>0}$ to $\mathbb{R}_{>0}$ such that for all $x, y > 0$, the two equations hold

$$\big(f(x)+y-1\big)\big(g(y)+x-1\big)=(x+y)^2,$$
$$\big(-f(x)+y\big)\big(g(y)+x\big)=(x+y+1)(y-x-1).$$

Note : $\mathbb{R}_{>0}$ denotes the set of positive real numbers.

5. Let r be a positive integer. Find the smallest positive integer m satisfying the condition : for all sets $A_1, A_2, \ldots, A_r$ with $A_i \cap A_j = \varnothing$, for all $i \neq j$, and $\bigcup_{k=1}^{r} A_k = \{1, 2, \ldots, m\}$, there exists $a, b \in A_k$ for some k such that $1 \leqslant \frac{b}{a} \leqslant 1 + \frac{1}{2022}$.

6. Does there exist positive integers $n_1, n_2, \ldots, n_{2022}$ such that the number

$$\left(n_1^{2020}+n_2^{2019}\right)\left(n_2^{2020}+n_3^{2019}\right)\cdots\left(n_{2021}^{2020}+n_{2022}^{2019}\right)\left(n_{2022}^{2020}+n_1^{2019}\right)$$

is a power of 11?

Time: 4 hours and 30 minutes
Each problem is worth 7 points