9 The PAMO in Maputo: April 2003

PAMO in Maputo 2003: Day 1
Time: 4.5 hours

1. Let $\mathbb{N}_0$ denote the set of nonnegative integers $\mathbb{N}_0 = \{0,\ 1,\ 2,\ 3,\ \dots\}$ Find all the functions $f : \mathbb{N}_0 \to \mathbb{N}_0$ satisfying the following three conditions:

   (i) $f(n) < f(n+1)$ for all $n \in \mathbb{N}_0$,

   (ii) $f(2) = 2$ and

   (iii) $f(mn) = f(m)f(n)$

   for all $m,\ n \in \mathbb{N}_0$.

2. The circumference of a circle is arbitrarily divided into four parts. The midpoints of the arcs are connected by line segments. Show that two of these segments are perpendicular.

3. Does there exist a base in which number of the form 10101, 101010101, 1010101010101, etc are all prime numbers?

PAMO in Maputo 2003: Day 2
Time: 4.5 hours

4. Let $\mathbb{N}_0$ denote the set of non negative integers $\mathbb{N}_0 = \{0,\ 1,\ 2,\ 3, \ldots,\}$. Does there exist a function $f : \mathbb{N}_0 \to \mathbb{N}_0$ such that, for all n:

$$f^{(2003)}(n) = 5n?$$

(Note: $f^{(2003)}$ means $f \circ f \circ f \circ \cdots \circ f$ 2003 times)

5. Find all positive integers $n \in \mathbb{N}$ such that 21 divides $2^{2^n} + 2^n + 1$.

6. Find all function $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x^2) - f(y^2) = (x+y)(f(x) - f(y))$$

for all x, y in $\mathbb{R}$.