First Day : 26th May 2010

Duration : 4 h 30

Instructions
- No calculating devices (computers, calculators, slide rules, etc), books, notes (printed or handwritten) are allowed in the examination room.
- Pens, pencils, rulers and compasses only may be used.

Exercise 1
a) Show that it is possible to pair off the numbers 1, 2, 3, … , 10 so that the sums of each of the five pairs are five different prime numbers.

b) Is it possible to pair off the numbers 1, 2, 3, … , 20 so that the sums of each of the ten pairs are ten different prime numbers?

Exercise 2
How many ways are there to line up 19 girls (all of different heights) in a row so that no girl has a shorter girl both in front of and behind her?

Exercise 3
In an acute-angled triangle ABC, CF is an altitude, with F on AB, and BM is a median, with M on CA. Given that BM = CF and \(<MBC = <FCA\), prove that triangle ABC is equilateral.
Exercise 4
Seven distinct points are marked on a circle of circumference $c$. Three of the points form an equilateral triangle and the other four form a square. Prove that at least one of the seven arcs into which the seven points divide the circle has length less than or equal to $\frac{c}{24}$.

Exercise 5
A sequence $a_0, a_1, a_2, \ldots, a_n, \ldots$ of positive integers is constructed as follows:
• if the last digit of $a_n$ is less than or equal to 5 then this digit is deleted and $a_{n+1}$ is the number consisting of the remaining digits. (If $a_{n+1}$ contains no digits the process stops.)
• otherwise $a_{n+1} = 9a_n$
Can one choose $a_0$ so that an infinite sequence is obtained?

Exercise 6
Does there exist a function $f : \mathbb{Z} \to \mathbb{Z}$ such that $f(x + f(y)) = f(x) - y$ for all integers $x$ and $y$?