25th PAN AFRICAN MATHEMATICS OLYMPIAD

Rabat from 1 to 7 July 2017

Day 1: Tuesday, July 4, 2017  Duration: 4 h 30 min

Problem 1

We consider the real sequence \((x_n)\) defined by \(x_0 = 0\), \(x_1 = 1\) and \(x_{n+2} = 3x_{n+1} - 2x_n\) for \(n = 0, 1, \ldots\). We define the sequence \((y_n)\) by \(y_n = x_n^2 + 2^{n+2}\) for every nonnegative integer \(n\). Prove that for every \(n > 0\), \(y_n\) is the square of an odd integer.

Problem 2

Let \(x\), \(y\) and \(z\) be positive real numbers such that \(xy + yz + zx = 3xyz\). Prove that \(x^2y + y^2z + z^2x \geq 2(x + y + z) - 3\). In which case do we have equality?

Problem 3

Let \(n\) be a positive integer. Find, in terms of \(n\), the number of pairs \((x, y)\) of positive integers that are solutions of the equation: \(x^2 - y^2 = 10^2 \cdot 30^{2n}\). Prove further that this number is never a square.
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Day 2 : Wednesday, july 5, 2017 Duration : 4 h 30 min

**Problem 4**

Find all the real numbers $x$ such that \[
\frac{1}{[x]} + \frac{1}{[2x]} = \{x\} + \frac{1}{3},
\]
where $[x]$ denotes the integer part of $x$ and $\{x\} = x - [x]$.

For example, $[2.5] = 2$, $\{2.5\} = 0.5$ and $[-1.7] = -2$, $\{-1.7\} = 0.3$.

**Problem 5**

The numbers from 1 to 2017 are written on a board. Deka and Farid play the following game : each of them, on his turn, erases one of the numbers. Anyone who erases a multiple of 2, 3 or 5 loses and the game is over. Is there a winning strategy for Deka?

**Problem 6**

Let $ABC$ be a triangle with $H$ its orthocenter. The circle with diameter $[AC]$ cuts the circumcircle of the triangle $ABH$ at $K$. Prove that the point of intersection of the lines $CK$ and $BH$ is the midpoint of the segment $[BH]$. 