U.M.A

Commission OPAM

25^{th} PAN AFRICAN MATHEMATICS OLYMPIAD

Rabat from 1 to 7 july 2017

Day 1 : Tuesday, july 4, 2017 **Duration : 4 h 30 min**

Problem 1

We consider the real sequence (x_n) defined by $x_0 = 0,\ x_1 = 1$ and $x_{n+2} = 3x_{n+1} - 2x_n$ for $n = 0, 1, \ldots$.
We define the sequence (y_n) by $y_n = x_n^2 + 2^{n+2}$ for every nonnegative integer n.
Prove that for every $n > 0$, y_n is the square of an odd integer.

Problem 2

Let $x,\ y$ and z be positive real numbers such that $xy + yz + zx = 3xyz$.
Prove that $x^2y + y^2z + z^2x \geqslant 2(x + y + z) - 3$.
In which case do we have equality ?

Problem 3

Let n be a positive integer. Find, in terms of n, the number of pairs (x, y) of positive integers that are solutions of the equation : $x^2 - y^2 = 10^2.30^{2n}$.
Prove further that this number is never a square.

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Day 2 : Wednesday, july 5, 2017 **Duration : 4 h 30 min**

Problem 4

Find all the real numbers x such that $\dfrac{1}{[x]} + \dfrac{1}{[2x]} = \{x\} + \dfrac{1}{3}$, where $[x]$ denotes the integer part of x and $\{x\} = x - [x]$.
For example, $[2.5] = 2,\ \{2.5\} = 0.5$ and $[-1.7] = -2,\ \{-1.7\} = 0.3$.

Problem 5

The numbers from 1 to 2017 are written on a board. Deka and Farid play the following game : each of them, on his turn, erases one of the numbers. Anyone who erases a multiple of $2, 3$ or 5 loses and the game is over. Is there a winning strategy for Deka ?

Problem 6

Let ABC be a triangle with H its orthocenter. The circle with diameter $[AC]$ cuts the circumcircle of the triangle ABH at K. Prove that the point of intersection of the lines CK and BH is the midpoint of the segment $[BH]$.